Fluxoids behavior in superconducting ladders

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Abstract. The nature of the interaction between fluxoids and between them and the external magnetic field is studied in one-dimensional superconducting networks. An Ising like expression is derived for the energy of a network revealing that fluxoids behave as repulsively interacting objects driven towards the network center by the effective applied field. Competition between these two interactions determines the equilibrium arrangement of fluxoids in the network as a function of the applied field. It is demonstrated that the fluxoids configurations are not always commensurate to the network symmetry. Incommensurate, degenerated configurations may be formed even in networks with an odd number of loops.

1. Introduction

The macroscopic quantum nature of superconductivity is manifested in loops, and generally in multiply-connected superconductors, in quantization of the 'fluxoid' defined as: $(4\pi\lambda^2/c)\oint j\cdot dl + \Phi$, where j is the shielding current in a closed loop, λ is the penetration depth, and Φ is the magnetic flux threading the loop [1]. In the early days of superconductivity, it was predicted by Fritz London [2], and later confirmed experimentally by Little and Parks [3], that the fluxoid must be an integer multiple of the flux quantum $\phi_0 = hc/2e$. Fluxoid quantization effects have been studied extensively, both theoretically and experimentally, in a variety of superconducting networks [4-12]. However, most of these studies adopt the mean field approach, providing no intuitive understanding of the interaction between fluxoids and the mechanism governing their arrangement in superconducting networks. The purpose of the present work is to elucidate the nature of the interaction between fluxoids and to clarify the physics behind their arrangements in networks. Analyzing the simplest case of a superconducting one-dimensional network ('ladder'), we show that fluxoids act as repulsively interacting objects dragged towards the ladder center by their interaction with the externally applied field. A competition between these two interactions determines the equilibrium positions of the fluxoids in the network as a function of the applied field. To demonstrate this concept we present calculated results of fluxoid arrangements in several examples of finite 1D and 2D networks.

2. Analysis

Our analysis is based on the "current squared" model (known as the "J^2 model") [6], In which the kinetic energy of the network is calculated as the sum of the squared currents over all the network wires. The number and arrangement of the fluxoids is determined by the requirement of minimum energy.

Consider a superconducting ladder of finite length, consisting of N square loops of unit side, as shown in Figure 1.

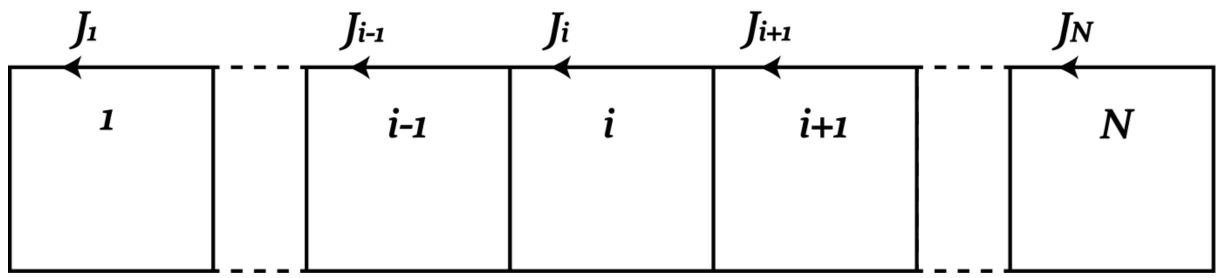

Figure 1: Finite ladder consisting of N square loops.

The fluxoid quantization equation for the loop i reads:

$$4J_i - J_{i-1} - J_{i+1} = n_i - \phi/\phi_0, \tag{1}$$

where n_i is the vorticity of the loop i, ϕ is the flux threading this loop, and by definition $J_i = 0$ for $0 > i$ or $i > N$. For simplicity, the coefficient $4\pi\lambda^2/c$ is taken as 1. According to Eq. (1), the set of the fluxoid quantization equations for all the loops can be written as a matrix equation:

$$\hat{A} \cdot \vec{J} = \vec{n} - \frac{\vec{\phi}}{\phi_0}, \tag{2}$$

where the elements of the matrix $\hat{A}$: $A_{ij} = 4\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}$, and $\delta_{i,j}$ is the Kronecker δ. The current vector $\vec{J}$ can be calculated from Eq. (2) by inversion:

$$\vec{J} = \hat{A}^{-1}(\vec{n} - \frac{\vec{\phi}}{\phi_0}). \tag{3}$$

Denoting the matrix $\hat{A}^{-1}$ as $\hat{B}$, Eq. (3) can be written as a set of equations:

$$J_i = \sum_{j=1}^{N} B_{ij}\left(n_j - \frac{\phi}{\phi_0}\right), \qquad i = 1..N. \tag{4}$$

Using the J^2 model, knowledge of J_i allows calculation of the energy E_i of the loop i:

$$E_i = 2J_i^2 + \frac{1}{2}(J_i - J_{i-1})^2 + \frac{1}{2}(J_i - J_{i+1})^2 + \frac{1}{2}J_1^2\delta_{1,i} + \frac{1}{2}J_N^2\delta_{N,i}$$
$$= J_i[3J_i - J_{i-1} - J_{i+1}] + \frac{1}{2}J_{i-1}^2 + \frac{1}{2}J_{i+1}^2 + \frac{1}{2}J_1^2\delta_{1,i} + \frac{1}{2}J_N^2\delta_{N,i},$$

and the total energy E of the network:

$$E = \sum_{i=1}^{N} E_i = \sum_{i=1}^{N}\left\{J_i(3J_i - J_{i-1} - J_{i+1}) + \frac{1}{2}J_{i-1}^2 + \frac{1}{2}J_{i+1}^2\right\} + \frac{1}{2}J_1^2 + \frac{1}{2}J_N^2 . \tag{5}$$

Using Eq. (1) and realizing that $\sum_{i=1}^{N}\left\{-J_i^2 + \frac{1}{2}J_{i-1}^2 + \frac{1}{2}J_{i+1}^2\right\} + \frac{1}{2}J_1^2 + \frac{1}{2}J_N^2 = 0$, Eq. (5) becomes

$$E = \sum_{i=1}^{N} J_i\left(n_i - \frac{\phi}{\phi_0}\right). \tag{6}$$

Inserting J_i from Eq. (4) yields

$$E = \sum_{i=1}^{N}\sum_{j=1}^{N} B_{ij}\left(n_j - \frac{\phi}{\phi_0}\right)\left(n_i - \frac{\phi}{\phi_0}\right) = \sum_{ij} B_{ij}\left(n_i n_j - 2\frac{\phi}{\phi_0}\, n_i + \left(\frac{\phi}{\phi_0}\right)^2\right). \tag{7}$$

The above expression for the total energy, E, is reminiscent of the Ising model for the energy of a spin configuration, having the form $\sum_{ij} J_{ij} S_i S_j$ - $\mu \sum_j h_j S_j$ [13]; n_j, and B_{ij} playing the role of the Ising spin S_j and the exchange interaction term J_{ij}, respectively. The first term on the right hand side of Eq. (7), $\left(\sum_{ij} B_{ij} n_i n_j\right)$, represents the interaction between fluxoids, including the self-interactions $\sum_i B_{ii} {n_i}^2$. The second term, $\left(-2\frac{\phi}{\phi_0}\sum_{ij} n_i B_{ij}\right)$, expresses the interaction between the fluxoids and the effective magnetic field. The third term, $(\phi/\phi_0)^2 \sum_{ij} B_{ij}$, is constant, independent of the vorticities, and thus may be ignored.

For the matrix $\hat{A}$ given in Eq. (2), $\hat{B} = \hat{A}^{-1}$ is a symmetric matrix with elements [14]:

$$B_{ij} = C\gamma^{i-j-1}\left(1 - \eta^{i}\right)\left(1 - \eta^{N+1-j}\right), \text{ for } i \le j. \tag{8}$$

where $\gamma = 2 + \sqrt{3}$, $\eta = (2 - \sqrt{3})/(2 + \sqrt{3})$ and $C = 1/(1 - \eta)(1 - \eta^{N+1})$. Due to the symmetry of $\hat{B}$, B_{ij} for $i > j$ can be calculated as B_{ji} using Eq. (8). Since $\eta \ll 1$, C is approximately 1 and B_{ij} can be approximated as $\gamma^{-(|i-j|+1)}$, for all i and j. Thus Eq. (7) becomes

$$E = \sum_{ij} \gamma^{-(|i-j|+1)}\left(n_i n_j - 2\frac{\phi}{\phi_0}\, n_i + \left(\frac{\phi}{\phi_0}\right)^2\right) . \tag{9}$$

The above expression shows that fluxoids can be treated as repulsively interacting objects, with interaction energy decreasing exponentially with their separation. In order to minimize the total energy, the repulsive interaction between fluxoids tends to keep them away from each other. On the other hand, the interaction between the fluxoids and the effective magnetic field, represented by the second term in Eq. (9), $\left(-2\phi/\phi_0 \sum_i n_i \sum_j {\gamma_1}^{-(|i-j|+1)}\right)$ tends to drive the fluxoids away from the network edges towards the network's center. This is because the effective magnetic field $-2(\phi/\phi_0)\sum_j {\gamma_1}^{-(|i-j|+1)} \propto \cosh\left(\ln(\gamma_1)\left((N+1)/2 - i\right)\right)$ is minimum at the center of the ladder $(i = (N+1)/2)$. Thus, while the external magnetic favors assembling the fluxoids near the ladder center, the fluxoids repel each other tending to keep themselves apart. Competition between these two opposite interactions determines the equilibrium arrangement of fluxoids in the network as a function of the applied field.

Considering the first fluxoid which enters the ladder, it always appears at the center of the network (or next to it, in a ladder with an even number of loops) as it is affected only by the external field which drives it to the center. As the field increases, a second fluxoid appears, pushing the first one out of its central position and both fluxoids arrange themselves in an optimum configuration around the center, keeping apart from each other and away from the network edges. The same principle determines the arrangements of the next fluxoids entering the ladder as the field further increases.

Our analysis of superconducting ladders can be extended to two dimensional superconducting networks. The basic idea that the fluxoid arrangements are determined by a competition between the fluxoid repulsive interaction and their interaction with the applied field, remains the same. However, computations of these interactions, and the resulting fluxoid equilibrium configurations in two dimensional networks become more complicated. The occupation process in 1D and 2D superconducting networks are demonstrated in the following section.

3. Simulations

As an example, we present calculated results for ladders with 7, 8 and 9 loops. In each case, the energy of the ladder, as a function of the loops vorticities and the external field, was calculated using Eq. (7) and the exact expression for the elements B_{ij} (Eq. 8). For each given field the fluxoid arrangement $(n_1, n_2, \ldots, n_N)$ which minimizes the energy was determined.

The solid, dashed, dashed-dotted curves in Figure 2 show the minimum energy as a function of the normalized flux ϕ/ϕ_0 in ladders with 7,8 and 9 loops, respectively. The crests in each curve indicate a change in the fluxoids configurations in the ladder. Thus, in the ladders with 7,8 and 9 elements the total number of configurations is 7,8 and 9, respectively. It is interesting to note that the number of configurations is not necessarily equal to the number of elements. For example, in ladders with rectangular loops attached along their long side, a change of the applied field can cause rearrangement of the same number of fluxoids, giving rise to an access number of configurations [15].

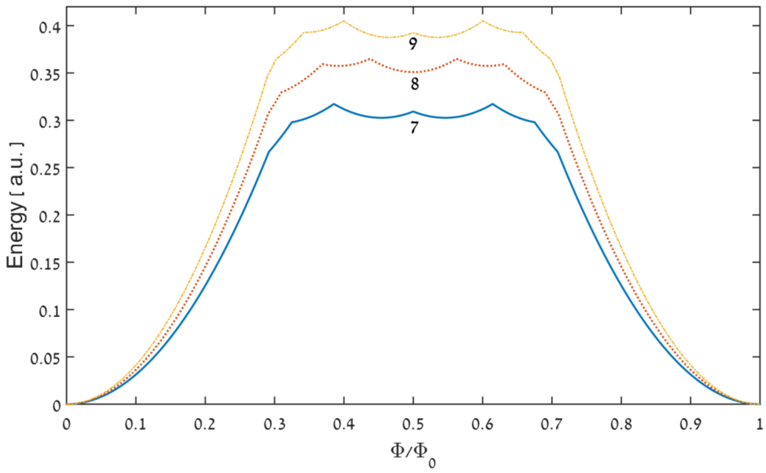

Figure 2: Energy as a function of the normalized magnetic flux in ladders with 7, 8 and 9 loops.

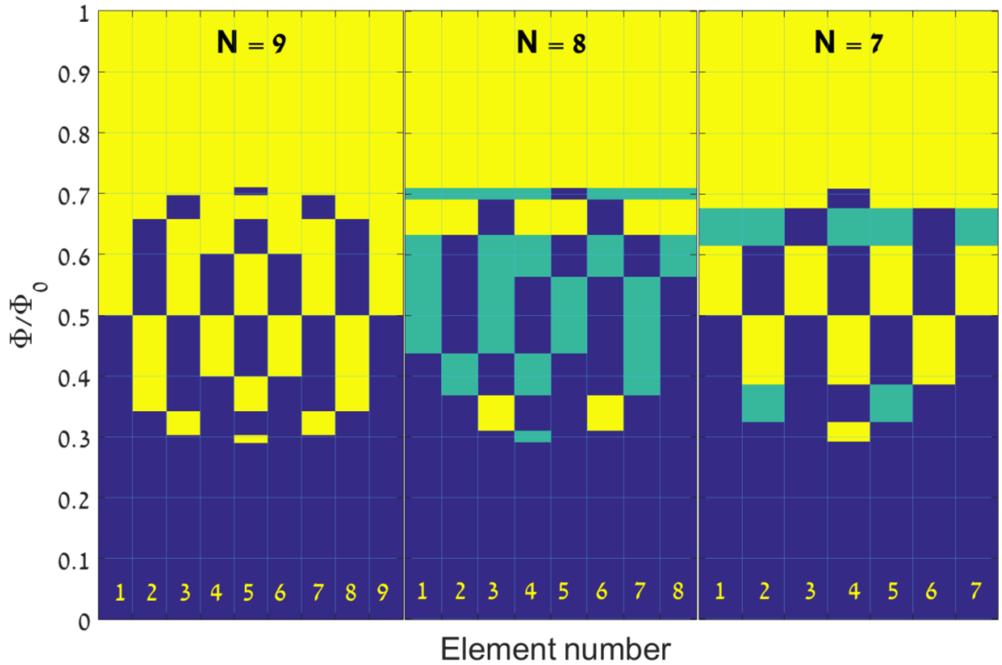

Figure 3: Fluxoid configuration as a function of magnetic flux in ladders with 7, 8 and 9 loops, in the first period. An empty loop is colored dark blue, and occupied loop is colored yellow or green. The green color indicates degenerated configurations which are incommensurate with the symmetry of the ladder.

The fluxoids arrangements as a function of field are illustrated in Figure 3. An empty loop is colored blue, and occupied loop is colored yellow or green. The green color indicates degenerated configurations which are incommensurate with the symmetry of the ladder. The ladders with 7 and 9 loops demonstrate the general rule that when the number of loops is odd, the first fluxoid always occupies the loop at the ladder's center. In ladders with an even number of loops, the first fluxoid occupies a degenerated state on either side of the center, as demonstrated by the ladder with 8 loops. As the field increases, a second fluxoid enters the ladder, pushing the first one out of its position and both fluxoids arrange themselves in an optimum configuration, keeping apart from each other and away from the network edges. In the ladder with 9 loops, this configuration conforms the symmetry of the ladder, however, this is not the case in the ladders with 7 and 8 elements. As more fluxoids enter the ladder with increasing field, rearrangement of fluxoids continues until the last fluxoid enters the ladder's center completing one period in which each loop is occupied with one fluxoid. Occupation of the loops in the following periods follows the same pattern.

Figure 4 illustrates calculated results for a 3x3 square network, based on the J^2 model. Note that the first fluxoid appears in the central loop of the network, as in a ladder. Also note that the number of different configurations (11) exceeds the number of loops in the network, due to rearrangement of the same number of fluxoids as the field increases. This situation occurs in configurations of 3 and 6 fluxoids. Among the 11 different configurations, there are 6 degenerated states that are incommensurate to the network symmetry (marked in green in Fig. 4). The degenerated configurations are obtained by applying the symmetry operations of the network. Thus, two degenerated configurations correspond to N=2,7, and four degenerated configurations correspond to each configuration with N=3,6.

We note that calculations based on the de-Gennes-Alexander equations for a network yield quite different results [16]. For example, following the appearance of the first fluxoid at the network central loop, the second fluxoid appears at the same loop creating a double fluxoid at the network center. This configuration has a higher energy than that the configuration of two separated fluxoids derived from the 'J^2 model' (see Fig. 4). Involving the appearance of anti-fluxoids in the network, the calculation based on the de-Gennes-Alexander equations, predicts 9 configurations all of which are commensurate to the network symmetry. It should be noted, however, that by minimizing the Ginzburg–Landau free energy, asymmetric fluxoid patterns have been reported for a 10x10 network [17]. Finally we note that in our calculations based on the J^2 model, the fluxoid configurations are temperature independent, as the only temperature dependent factor in this model is $4\pi\lambda^2/c$ which scales the current density, and the square of this factor scales the energy.

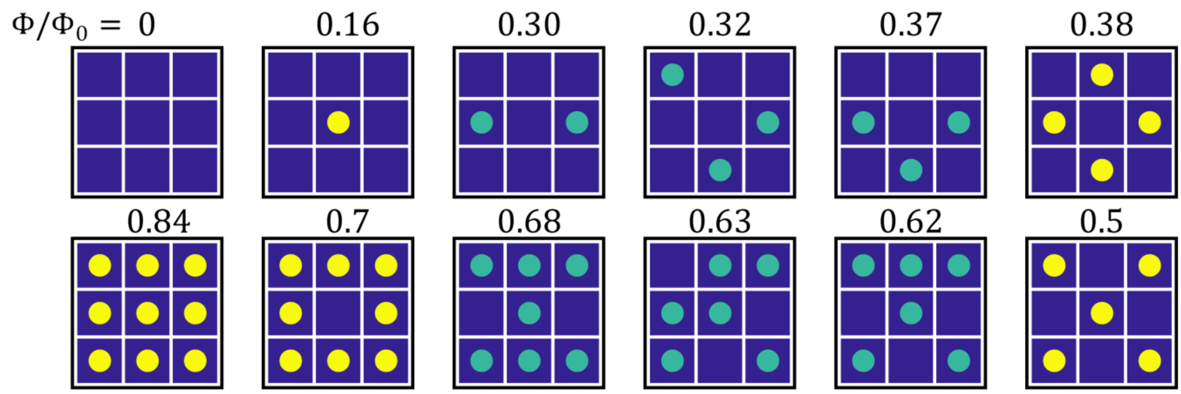

Figure 4: Fluxoid configurations in 3x3 square network calculated in the framework of the 'J^2 model'. An empty loop is colored dark blue, and occupied loop is colored yellow or green. The green color indicates degenerated configurations which are incommensurate with the symmetry of the ladder.

4. Summary and conclusions

An Ising-like expression derived for the energy of fluxoids in a 1D superconducting network reveals that the fluxoids act as repulsively interacting objects with an interaction energy that decreases exponentially with their relative separation. In this expression, the effective magnetic field drives the fluxoids toward the network center. The competition between these two interactions determines the equilibrium configuration of the fluxoids in the ladders. These configurations may be incommensurate to the symmetry of the ladder, in ladders with even as well as odd number of loops. Fluxoids in 2D networks follow a similar pattern, i.e. they repel each other and are driven to the center by the applied field.

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